Essential spectra and exponential estimates of eigenfunctions of lattice operators of quantum mechanics

2009 J. Phys. A: Math. Theor. 42 385207

(http://iopscience.iop.org/1751-8121/42/38/385207)

J. Phys. A: Math. Theor. **42** (2009) 385207 (21pp) doi:10.1088/1751-8113/42/38/385207

Essential spectra and exponential estimates of eigenfunctions of lattice operators of quantum mechanics

Vladimir S Rabinovich[1] **and Steffen Roch**[2]

[1] Instituto Politécnico Nacional, ESIME-Zacatenco, Av. IPN, edif. 1, México D.F., 07738, México, Mexico
[2] Fachbereich Mathematik, Technische Universität Darmstadt, Schlossgartenstrasse 7, 64289 Darmstadt, Germany

E-mail: vladimir_rabinovich@hotmail.com and roch@mathematik.tu-darmstadt.de

Received 19 January 2009, in final form 14 May 2009
Published 7 September 2009
Online at stacks.iop.org/JPhysA/42/385207

Abstract
This paper is devoted to estimates of the exponential decay of eigenfunctions of difference operators on the lattice $\mathbb{Z}^n$ which are discrete analogs of the Schrödinger, Dirac and square-root Klein–Gordon operators. Our investigation of the essential spectra and the exponential decay of eigenfunctions of the discrete spectra is based on the calculus of pseudodifference operators (i.e., pseudodifferential operators on the group $\mathbb{Z}^n$ with analytic symbols), and the limit operators method. We obtain a description of the location of the essential spectra and estimates of the eigenfunctions of the discrete spectra of the main lattice operators of quantum mechanics, namely: matrix Schrödinger operators on $\mathbb{Z}^n$, Dirac operators on $\mathbb{Z}^3$ and square root Klein–Gordon operators on $\mathbb{Z}^n$.

PACS numbers: 63.10.+a, 63.22.Gh
Mathematics Subject Classification: 39A47, 47B39, 81Q10

1. Introduction

Exponential estimates of solutions of elliptic partial differential equations in general, and of the Schrödinger equation in particular, are a classical and central topic of analysis. There is an extensive bibliography devoted to this problem (see [1, 2, 10, 12–14], for instance). Exponential estimates of solutions of pseudodifferential equations are considered in [22, 27, 28, 31, 34, 35]. In [40, 41], the authors proposed a new approach to exponential estimates for partial differential and pseudodifferential operators which is based on the limit operators method, as developed in [42].

We consider difference operators of the form

$$A = \sum_{\alpha \in M} a_\alpha V_\alpha \tag{1}$$

acting on the space $l^2(\mathbb{Z}^n, \mathbb{C}^N)$ of squared integrable functions on the lattice $\mathbb{Z}^n$ with values in $\mathbb{C}^N$. In (1), M is a finite subset of $\mathbb{Z}^n$, the a_α refer to operators of multiplication by matrix-valued functions in $l^\infty(\mathbb{Z}^n, \mathbb{C}^{N\times N})$, and $(V_\alpha u)(x) = u(x-\alpha)$ is the operator of shift by $\alpha \in \mathbb{Z}^n$. The main aim of the present paper is the relation between the location of the essential spectrum of the operator A and estimates of the exponential decay of eigenfunctions of discrete spectrum of the operator A. The essential spectrum of operators of the form (1) and of more general operators, belonging to the Wiener algebra on $\mathbb{Z}^n$, was examined by the authors in the book [42] by means of the so-called limit operators method, see also the related papers [37–39].

Spectral problems for difference operators (1) arise in many physical problems. We will focus our attention on a model from solid state physics, viz. the harmonic vibrations of atoms of infinite crystals (phonons). First consider the cubic crystal modeled by the lattice $\mathbb{Z}^n$ (for details see [5, chapter 22], [8, chapter 5] and [23, chapter 5]).

Let $u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t))$ denote the deviation of the particle located at $x \in \mathbb{Z}^n$ from its equilibrium position at the moment t. The equation of harmonic vibrations of the atomic lattice $\mathbb{Z}^n$ can be written as

$$m\ddot{u}(x,t) = -\sum_{\gamma \in \Gamma} a_\gamma(x)(V_\gamma u)(x,t) + a_0(x)u(x,t), \qquad t \in \mathbb{R}, x \in \mathbb{Z}^n, \tag{2}$$

where Γ is the set of the vectors $\pm\gamma_j := (0, \ldots, \pm 1, \ldots, 0)$ with ± 1 standing at the jth position and $j = 1, \ldots, n$. Further, $m > 0$ is the atomic mass of the particles, $a_\gamma(x) = \left(a_\gamma^{ij}(x)\right)_{i,j=1}^3$ is the matrix of interaction between the atom located at the point x and its adjacent atoms at the points $x \pm \gamma_k$ for $k = 1, \ldots, n$, and $a_0(x) := \operatorname{diag}(a_0^1(x), a_0^2(x), a_0^3(x))$ is a diagonal matrix describing the external forces acting on the particle x. We suppose that m, a_γ^{ij} and a_j belong to $l^\infty(\mathbb{Z}^n)$. Under the conditions $a_{-\gamma}^{ij}(x) = \bar{a}_\gamma^{ji}(x+\gamma)$, the a_0^j are real-valued functions, and the operator

$$A = -\sum_{\gamma \in \Gamma} a_\gamma V_\gamma + a_0 \tag{3}$$

is a self-adjoint operator on a Hilbert space $l^2(\mathbb{Z}^n, \mathbb{C}^3) = l^2(\mathbb{Z}^n) \otimes \mathbb{C}^3$. As usual, we will seek solutions of equation (2) of the form $u(x,t) = v(x)\mathrm{e}^{-\mathrm{i}\omega t}$. For the definition of the eigen-frequencies ω and normal modes v_ω of equation (3) we consider the spectral equation

$$Av = -\lambda v, \qquad \lambda = m\omega^2. \tag{4}$$

If m, a_γ and a_0 are independent of x, then the operator A does not have eigenvalues. Then the spectrum of A is continuous, and it is given by the dispersion equation

$$\det\left(\sum_{\gamma \in \Gamma} a_\gamma \,\mathrm{e}^{\mathrm{i}\gamma\cdot\xi} - a_0 - \lambda E_3\right) = 0, \qquad \xi \in [0, 2\pi]^3, \qquad \lambda = m\omega^2.$$

If the matrices a_0 and a_γ depend on $x \in \mathbb{Z}^n$, then the spectral problem (4) turns out to be much more complicated, since now the spectral properties of the operator A depend essentially on the structure of the matrices a_γ and a_0. In particular, there may be a finite or infinite sequence of eigen-frequencies $\omega_1, \omega_2, \ldots$ with corresponding normal modes $v_1(x), v_2(x), \ldots$. Hence, in this case, equation (2) has solutions of the form $u_j(x,t) = \mathrm{e}^{-\mathrm{i}\omega_j t} v_j(x)$ with $v_j \in l^2(\mathbb{Z}^n, \mathbb{C}^3)$.

We shall see that the v_j are actually exponentially decreasing at infinity and we derive estimates which describe the decrease.

In the simplest case of an isotropic interaction matrix of the form $a_\gamma^{ij}(x) = b(x)\delta^{ij}$ where $\delta^{ij} = 1$ if $i = j$ and $\delta^{ij} = 0$ if $i \neq j$ (= the Kronecker matrix), the operator A is a diagonal operator on $l^2(\mathbb{Z}^n, \mathbb{R}^3)$ of the form

$$A = \left(b \sum_{j=1}^{n} (V_{\gamma_j} + V_{-\gamma_j}) + a_0 \right) E_3 \tag{5}$$

with real-valued functions b and a_0 and 3×3-identity matrix E_3. The operator (5) can be viewed as the lattice analog of the Schrödinger operator on $\mathbb{R}^n$. Operators of this kind arise in many other physical problems, for instance, in the tight-binding approximation in solid state physics (see, for instance, [9, 21, 29, 30]), in the Andersen tight binding localization problems (see [15, 16, 47, 49] and others), and in the investigation of spectral properties of carbon nanostructure (see [17] and the literature cited there). Different aspects of the spectral theory of discrete Schrödinger operators are also considered in [3, 4, 19, 20, 47, 53, 54].

Previously, discrete Dirac operators also attracted much attention. They were used, e.g., in comparative studies of relativistic and nonrelativistic electron localization phenomena [6], in relativistic investigations of electrical conduction in disordered systems [46], in the construction of supertransparent models with supersymmetric structures [50], and in relativistic tunneling problems [45].

Our approach to study essential spectra and the exponential decay of eigenfunctions is based on the calculus of pseudodifference operators (i.e., pseudodifferential operators on the group $\mathbb{Z}^n$) with analytic symbols as developed in [37], and the limit operators method (see [42] and the references cited there).

The paper is organized as follows. In section 2 we recall some auxiliary facts on the pseudodifference operators with analytic symbols on $\mathbb{Z}^n$, limit operators, essential spectra and the behavior of solutions of pseudodifference equations at infinity.

In section 3 we consider the discrete Schrödinger operators on $l^2(\mathbb{Z}^n, \mathbb{C}^N)$ of the form

$$(Hu)(x) = \sum_{k=1}^{n} \left(V_{e_k} - \mathrm{e}^{\mathrm{i}a_k(x)}\right)\left(V_{-e_k} - \mathrm{e}^{-\mathrm{i}a_k(x)}\right)u(x) + \Phi(x)u(x),$$

where V_{e_k} is the operator of shift by e_k, the a_k are real-valued bounded slowly oscillating functions on $\mathbb{Z}^n$, and Φ is a Hermitian slowly oscillating and bounded matrix function on $\mathbb{Z}^n$. We show that the essential spectrum $\mathrm{sp}_{\mathrm{ess}}\, H$ of H is the interval

$$\mathrm{sp}_{\mathrm{ess}} H = \bigcup_{j=1}^{n} \left[\lambda_j^{\inf}, \lambda_j^{\sup} + 4n\right],$$

where

$$\lambda_j^{\inf} := \liminf_{x\to\infty} \lambda_j(\Phi(x)), \qquad \lambda_j^{\sup} := \limsup_{x\to\infty} \lambda_j(\Phi(x))$$

and where $\lambda_j(\Phi(x))$ are the increasingly ordered eigenvalues of the matrix $\Phi(x)$, i.e.

$$\lambda_1(\Phi(x)) < \lambda_2(\Phi(x)) < \cdots < \lambda_N(\Phi(x))$$

for $x \in \mathbb{Z}^n$ large enough. Note that $\mathrm{sp}_{\mathrm{ess}}\, H$ does not depend on the exponents a_k, and that there is a gap $\left(\lambda_j^{\sup} + 4n, \lambda_{j+1}^{\inf}\right)$ in the essential spectrum of H if $\lambda_j^{\sup} + 4n < \lambda_{j+1}^{\inf}$.

We also obtain the following estimates of eigenfunctions belonging to points in the discrete spectrum of H. In each of the cases

- $\lambda \in \left(\lambda_j^{\sup} + 4n, \lambda_{j+1}^{\inf}\right)$ is an eigenvalue of H and

$$0 < r < \cosh^{-1}\left(\frac{\min\left\{\lambda - \lambda_j^{\sup} - 2n, \lambda_{j+1}^{\inf} - \lambda + 2n\right\}}{2n}\right),$$

- $\lambda > \lambda_N^{\sup} + 4n$ is an eigenvalue of H and

$$0 < r < \cosh^{-1}\left(\frac{\lambda - \lambda_N^{\sup} - 2n}{2n}\right),$$

- $\lambda < \lambda_1^{\inf}$ is an eigenvalue of H and

$$0 < r < \cosh^{-1}\left(\frac{\lambda_1^{\inf} - \lambda + 2n}{2n}\right),$$

every λ-eigenfunction u of H has the property that $e^{r|x|}u \in l^p(\mathbb{Z}^n, \mathbb{C}^N)$ for every $1 < p < \infty$.

In section 4 we introduce self-adjoint Dirac operators on the lattice $\mathbb{Z}^3$ with variable slowly oscillating electric potentials. In accordance with the general properties of Dirac operators on $\mathbb{R}^3$ (see for instance [7, 51]), the corresponding discrete Dirac operator on $\mathbb{Z}^3$ should be a self-adjoint system of first-order difference operators. We are going to construct three-dimensional Dirac operators with this property following an idea proposed in [32, 33] for the construction of Dirac operators on $\mathbb{Z}$. Thus, we let

$$\mathcal{D} := \mathcal{D}_0 + e\Phi E_4, \tag{6}$$

where

$$\mathcal{D}_0 := c\hbar d_k \gamma^k + c^2 m \gamma^0,$$

E_N is the $N \times N$ unit matrix, the γ^k with $k = 0, 1, 2, 3$ refer to the 4×4 Dirac matrices, the

$$d_k := I - V_{e_k}, \qquad k = 1, 2, 3$$

are difference operators of the first order, $\hbar$ is Planck's constant, c is the speed of light, m and e are the mass and the charge of the electron and, finally, Φ is the real electric potential. The operator $\mathcal{D}$, acting on $l^2(\mathbb{Z}^3, \mathbb{C}^4)$, can be considered as the direct discrete analog of the Dirac operator on $\mathbb{R}^3$, but note that $\mathcal{D}$ is not *self-adjoint* on $l^2(\mathbb{Z}^3, \mathbb{C}^4)$. To force the self-adjointness, we consider the 'symmetrization' $\mathbb{D} := \mathbb{D}_0 + e\Phi I$ of $\mathcal{D}$ with

$$\mathbb{D}_0 := \begin{pmatrix} 0 & \mathcal{D}_0 \\ \mathcal{D}_0^* & 0 \end{pmatrix}, \tag{7}$$

which acts on $l^2(\mathbb{Z}^3, \mathbb{C}^8)$. The operator $\mathbb{D}$ is self-adjoint, and

$$\mathbb{D}_0^2 = \begin{pmatrix} (\hbar^2 c^2 \Gamma + m^2 c^4) E_4 & 0 \\ 0 & (\hbar^2 c^2 \Gamma + m^2 c^4) E_4 \end{pmatrix},$$

where $\hbar^2 c^2 \Gamma + m^2 c^4$ is the lattice Klein–Gordon Hamiltonian with Laplacian

$$\Gamma := \sum_{k=1}^{3} d_k^* d_k = \sum_{k=1}^{3} (2I - V_{e_k} - V_{e_k}^*).$$

We prove that the essential spectrum of $\mathbb{D}$ is the union

$$\begin{aligned} \mathrm{sp}_{\mathrm{ess}}\mathbb{D} = &\left[e\Phi^{\inf} - \sqrt{12\hbar^2 c^2 + m^2 c^4}, e\Phi^{\sup} - mc^2\right] \\ &\cup \left[e\Phi^{\inf} + mc^2, e\Phi^{\sup} + \sqrt{12\hbar^2 c^2 + m^2 c^4}\right], \end{aligned}$$

where

$$\Phi^{\inf} := \liminf_{x\to\infty} \Phi(x), \qquad \Phi^{\sup} := \limsup_{x\to\infty} \Phi(x).$$

Again we observe that if $e\Phi^{\sup} - e\Phi^{\inf} < 2mc^2$, then the essential spectrum of $\mathbb{D}$ has the gap $(e\Phi^{\sup} - mc^2, e\Phi^{\inf} + mc^2)$.

We also obtain the following estimates of eigenfunctions of the discrete spectrum. Let λ be a point of the discrete spectrum, and let λ and $r > 0$ satisfy one of the conditions

- $\lambda \in (e\Phi^{\sup} - mc^2, e\Phi^{\inf} + mc^2)$ and
$$0 < r < \cosh^{-1}\left(\frac{m^2c^4 - \max\{(e\Phi^{\inf} - \lambda)^2, (e\Phi^{\sup} - \lambda)^2\} + 6\hbar^2c^2}{6\hbar^2c^2}\right);$$
- $\lambda > e\Phi^{\sup} + \sqrt{12\hbar^2c^2 + m^2c^4}$ and
$$0 < r < \cosh^{-1}\left(\frac{(e\Phi^{\sup} - \lambda)^2 - m^2c^4 - 6\hbar^2c^2}{6\hbar^2c^2}\right);$$
- $\lambda < e\Phi^{\inf} - \sqrt{12\hbar^2c^2 + m^2c^4}$ and
$$0 < r < \cosh^{-1}\left(\frac{(e\Phi^{\inf} - \lambda)^2 - m^2c^4 - 6\hbar^2c^2}{6\hbar^2c^2}\right).$$

Then every λ-eigenfunction u of the operator $\mathbb{D}$ satisfies $\mathrm{e}^{r|x|}u \in l^p(\mathbb{Z}^3, \mathbb{C}^8)$ for every $p \in (1, \infty)$.

In section 5, we consider the lattice model of the relativistic square root Klein-Gordon operator as the pseudodifference operator of the form

$$\mathcal{K} := \sqrt{c^2\hbar^2\Gamma + m^2c^4} + e\Phi$$

on $l^2(\mathbb{Z}^n)$. We determine the essential spectrum of $\mathcal{K}$ and obtain exact estimates of the exponential decay at infinity of eigenfunctions of the discrete spectrum.

2. Pseudodifference operators, essential spectra, and exponential estimates

2.1. Some function spaces

For each Banach space X, $\mathcal{B}(X)$ refers to the Banach algebra of all bounded linear operators acting on X. For $1 \leqslant p \leqslant \infty$, we let $l^p(\mathbb{Z}^n, \mathbb{C}^N)$ denote the Banach space of all functions on $\mathbb{Z}^n$ with values in $\mathbb{C}^N$ with the norm

$$\|f\|^p_{l^p(\mathbb{Z}^n,\mathbb{C}^N)} := \sum_{x\in\mathbb{Z}^n} \|f(x)\|^p_{\mathbb{C}^N} < \infty \quad \text{if } p < \infty,$$

$$\|f\|_{l^\infty(\mathbb{Z}^n,\mathbb{C}^N)} := \sup_{x\in\mathbb{Z}^n} \|f(x)\|_{\mathbb{C}^N} < \infty.$$

The choice of the norm on $\mathbb{C}^N$ is not of importance in general; only for $p = 2$ we choose the Euclidean norm (such that $l^2(\mathbb{Z}^n, \mathbb{C}^N)$ becomes a Hilbert space and $\mathcal{B}(\mathbb{C}^N)$ a C^*-algebra in the usual way). Given a positive function w on $\mathbb{Z}^n$, which we will call a weight, let $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ stand for the Banach space of all functions on $\mathbb{Z}^n$ with values in $\mathbb{C}^N$ such that

$$\|u\|_{l^p(\mathbb{Z}^n,\mathbb{C}^N,w)} := \|wu\|_{l^p(\mathbb{Z}^n,\mathbb{C}^N)} < \infty.$$

Similarly, we write $l^\infty(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$ for the Banach algebra of all bounded functions on $\mathbb{Z}^n$ with values in $\mathcal{B}(\mathbb{C}^N)$ and the norm

$$\|f\|_{l^\infty(\mathbb{Z}^n,\mathcal{B}(\mathbb{C}^N))} := \sup_{x\in\mathbb{Z}^n} \|f(x)\|_{\mathcal{B}(\mathbb{C}^N)} < \infty.$$

Finally, we call a function $a \in l^\infty(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$ *slowly oscillating* if

$$\lim_{x\to\infty} \|a(x+y) - a(x)\|_{\mathcal{B}(\mathbb{C}^N)} = 0$$

for every point $y \in \mathbb{Z}^n$. We denote the class of all slowly oscillating functions by $SO(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$ and write simply $SO(\mathbb{Z}^n)$ in case $N = 1$.

2.2. Pseudodifference operators

Consider the n-dimensional torus $\mathbb{T}^n$ as a multiplicative group and let

$$\mathrm{d}\mu := \left(\frac{1}{2\pi \mathrm{i}}\right)^n \frac{\mathrm{d}t_1 \cdot \ldots \cdot \mathrm{d}t_n}{t_1 \cdot \ldots \cdot t_n} = \left(\frac{1}{2\pi \mathrm{i}}\right)^n \frac{\mathrm{d}t}{t}$$

denote the corresponding normalized Haar measure on $\mathbb{T}^n$.

Definition 1. *Let $\mathcal{S}(N)$ denote the class of all functions $a : \mathbb{Z}^n \times \mathbb{T}^n \to \mathcal{B}(\mathbb{C}^N)$ with*

$$\|a\|_k := \sup_{(x,t)\in\mathbb{Z}^n\times\mathbb{T}^n, |\alpha|\leqslant k} \left\|\partial_t^\alpha a(x,t)\right\|_{\mathcal{B}(\mathbb{C}^N)} < \infty \tag{8}$$

for every non-negative integer k, provided with the convergence defined by the semi-norms $|a|_k$. To each function $a \in \mathcal{S}(N)$, we associate the pseudodifference operator

$$(\mathrm{Op}\,(a)u)(x) := \int_{\mathbb{T}^n} a(x,t)\hat{u}(t)t^x \,\mathrm{d}\mu(t), \qquad x \in \mathbb{Z}^n, \tag{9}$$

which is defined on vector-valued functions with finite support. Here, $\hat{u}$ refers to the discrete Fourier transform of u, i.e.,

$$\hat{u}(t) := \sum_{x\in\mathbb{Z}^n} u(x)t^x, \qquad t \in \mathbb{T}^n.$$

We denote the class of all pseudodifference operators by $OP\mathcal{S}(N)$.

Pseudodifference operators on $\mathbb{Z}^n$ can be thought of as the discrete analog of pseudodifferential operators on $\mathbb{R}^n$ (see for instance [48, 52]); they can be also interpreted as (abstract) pseudodifferential operators with respect to the group $\mathbb{Z}^n$. For another representation of pseudodifference operators, we need the operator V_α of shift by $\alpha \in \mathbb{Z}^n$, i.e. the operator V_α on $l^p(\mathbb{Z}^n, \mathbb{C}^N)$ which acts via

$$(V_\alpha u)(x) = u(x-\alpha), \qquad x \in \mathbb{Z}^n.$$

Then the operator $\mathrm{Op}\,(a)$ can be written as

$$\mathrm{Op}\,(a) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha V_\alpha, \tag{10}$$

where

$$a_\alpha(x) := \int_{\mathbb{T}^n} a(x,t)t^\alpha \,\mathrm{d}\mu(t).$$

Integrating by parts we obtain

$$\|a_\alpha\|_{l^\infty(\mathbb{Z}^n,\mathcal{B}(\mathbb{C}^N))} \leqslant C|a|_2(1+|\alpha|)^{-2}, \tag{11}$$

whence

$$\|\mathrm{Op}\,(a)\|_{W(\mathbb{Z}^n,\mathbb{C}^N)} := \sum_{\alpha\in\mathbb{Z}^n} \|a_\alpha\|_{l^\infty(\mathbb{Z}^n,\mathcal{B}(\mathbb{C}^N))} < \infty. \tag{12}$$

We thus obtain that the pseudodifference operator $\mathrm{Op}\,(a)$ belongs to the Wiener algebra $W(\mathbb{Z}^n, \mathbb{C}^N)$ which, by definition, consists of all operators of the form (10) with norm (12). It is an immediate consequence of this fact that all operators $\mathrm{Op}\,(a)$ in $OP\mathcal{S}(N)$ are bounded on $l^p(\mathbb{Z}^n, \mathbb{C}^N)$ for all $p \in [1, \infty]$. Moreover, since the algebra $W(\mathbb{Z}^n, \mathbb{C}^N)$ is inverse closed in $\mathcal{B}(l^p(\mathbb{Z}^n, \mathbb{C}^N))$, the spectrum of $\mathrm{Op}\,(a) \in OP\mathcal{S}(N)$ is independent of the underlying space $l^p(\mathbb{Z}^n, \mathbb{C}^N)$. For details on the Wiener algebra and pseudodifference operators, see sections 2.5 and 5.1 in [42]. Also the following facts can be found there.

The operator (9) can be also written as

$$\mathrm{Op}(a)u(x) = \sum_{y\in\mathbb{Z}^n} \int_{\mathbb{T}^n} a(x, t)t^{x-y}u(y)\,\mathrm{d}\mu(t),$$

which leads to the following generalization of pseudodifference operators. Let a be a function on $\mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{T}^n$ with values in $\mathcal{B}(\mathbb{C}^N)$ which is subject to the estimates

$$|a|_k =: \sup_{(x,y,t)\in\mathbb{Z}^n\times\mathbb{Z}^n\times\mathbb{T}^n, |\alpha|\leqslant k} \left\|\partial_t^\alpha a(x, y, t)\right\|_{\mathcal{B}(\mathbb{C}^N)} < \infty \tag{13}$$

for every non-negative integer k. Let $\mathcal{S}_d(N)$ denote the set of all functions with these properties. To each function $a \in \mathcal{S}_d(N)$, we associate the *pseudodifference operator with double symbol*

$$(\mathrm{Op}_d\,(a)u)(x) := \sum_{y\in\mathbb{Z}^n} \int_{\mathbb{T}^n} a(x, y, t)u(y)t^{x-y}\,\mathrm{d}\mu(t), \tag{14}$$

where $u : \mathbb{Z}^n \to \mathbb{C}^N$ is a function with finite support. The right-hand side of (14) has to be understood as in (5.6) in [42], which is in analogy with the definition of an oscillatory integral (see [48] and also section 4.1.2 in [42]). The class of all operators of this form is denoted by $OP\mathcal{S}_d(N)$.

The representation of operators on $\mathbb{Z}^n$ as pseudodifference operators is very convenient due to the fact that one has explicit formulae for products and adjoints of such operators. The basic results are as follows (see propositions 5.1.4, 5.1.5 and 5.1.7 in [42]).

Proposition 1. *(i) Let $a, b \in \mathcal{S}(N)$. Then the product* $\mathrm{Op}\,(a)\mathrm{Op}\,(b)$ *is an operator in* $OP\mathcal{S}(N)$*, and* $\mathrm{Op}\,(a)\mathrm{Op}\,(b) = \mathrm{Op}\,(c)$ *with*

$$c(x, t) = \sum_{y\in\mathbb{Z}^n} \int_{\mathbb{T}^n} a(x, t\tau)b(x + y, \tau)\tau^{-y}\,\mathrm{d}\mu(\tau), \tag{15}$$

with the right-hand side understood as an oscillatory integral.

(ii) Let $a \in \mathcal{S}(N)$ and consider $\mathrm{Op}\,(a)$ *as acting on $l^p(\mathbb{Z}^n, \mathbb{C}^N)$ with $p \in (1, \infty)$. Then the adjoint operator of* $\mathrm{Op}\,(a)$ *belongs to* $OP\mathcal{S}(N)$*, too, and it is of the form* $\mathrm{Op}\,(a)^* = \mathrm{Op}\,(b)$ *with*

$$b(x, t) = \sum_{y\in\mathbb{Z}^n} \int_{\mathbb{T}^n} a^*(x + y, t\tau)\tau^{-y}\,\mathrm{d}\mu(\tau), \tag{16}$$

where $a^*(x, t)$ is the usual adjoint (i.e., transposed and complex conjugated) matrix.

(iii) Let $a \in \mathcal{S}_d(N)$. Then $\mathrm{Op}_d\,(a) \in OP\mathcal{S}(N)$*, and* $\mathrm{Op}_d\,(a) = \mathrm{Op}\,(a^\#)$ *where*

$$a^\#(x, t) = \sum_{y\in\mathbb{Z}^n} \int_{\mathbb{T}^n} a(x + y, t\tau)\tau^{-y}\mathrm{d}\mu(\tau).$$

2.3. Limit operators and the essential spectrum

Recall that an operator $A \in \mathcal{B}(X)$ is a *Fredholm operator* if its kernel $\ker A = \{x \in X : Ax = 0\}$ and its cokernel $\operatorname{coker} A = X/(AX)$ are finite-dimensional linear spaces. The essential spectrum of A consists of all points $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ is not a Fredholm operator. We denote the (usual) spectrum and the essential spectrum of A by $\operatorname{spec}_X A$ and $\operatorname{sp}_{\mathrm{ess}\, X} A$, respectively.

Our main tool to study the Fredholm property is limit operators. The following definition is crucial in what follows.

Definition 2. *Let $A \in \mathcal{B}(l^p(\mathbb{Z}^n, \mathbb{C}^N))$ with $p \in (1, \infty)$, and let $h : \mathbb{N} \to \mathbb{Z}^n$ be a sequence which tends to infinity in the sense that $|h(n)| \to \infty$ as $n \to \infty$. An operator $A^h \in \mathcal{B}(l^p(\mathbb{Z}^n, \mathbb{C}^N))$ is called a limit operator of A with respect to the sequence h if*

$$\text{s-lim}_{m\to\infty} V_{-h(m)} A V_{h(m)} = A^h \quad \textit{and} \quad \text{s-lim}_{m\to\infty} V_{-h(m)} A^* V_{h(m)} = (A^h)^*,$$

where s-lim *refers to the strong limit. Clearly, every operator has at most one limit operator with respect to a given sequence. We denote the set of all limit operators of A by $op(A)$.*

Let aI be the operator of multiplication by the function $a \in l^\infty(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$. A standard Cantor diagonal argument shows that every sequence h tending to infinity possesses a subsequence g such that, for every $x \in \mathbb{Z}^n$, the limit

$$\lim_{m\to\infty} a(x + g(m)) =: a^g(x)$$

exists. Clearly, a^g is again in $l^\infty(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$. Hence, all limit operators of aI are of the form $a^g I$. In particular, if $a \in SO(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$, then it follows easily from the definition of a slowly oscillating function that all limit operators of aI are of the form $a^g I$ where now $a^g \in \mathcal{B}(\mathbb{C}^N)$ is a constant function.

Let $\operatorname{Op}(a) \in OP\mathcal{S}(N)$, and let $h : \mathbb{N} \to \mathbb{Z}^n$ be a sequence tending to infinity. Then $V_{-h(m)} A V_{h(m)} = \operatorname{Op}(a_m)$ with $a_m(x) := a(x + h(m), t)$. It follows as above that the sequence h has a subsequence g such that $a(x + g(m), t)$ converges to a limit $a^g(x, t)$ for every $x \in \mathbb{Z}^n$ uniformly with respect to $t \in \mathbb{T}^n$. One can prove that the so-defined function a^g belongs to $\mathcal{S}(N)$ and the associated operator $\operatorname{Op}(a^g)$ is the limit operator of $\operatorname{Op}(a)$ with respect to g.

The following theorem, which is theorem 5.2.3 in [42], gives a complete description of the essential spectrum of pseudodifference operators in terms of their limit operators.

Theorem 1. *Let $a \in \mathcal{S}(N)$. Then, for every $p \in (1, \infty)$,*

$$\operatorname{sp}_{\mathrm{ess}\, l^p} \operatorname{Op}(a) = \bigcup_{\operatorname{Op}(a^g) \in op(A)} \operatorname{spec}_{l^r} \operatorname{Op}(a^g) \tag{17}$$

where $r \in [1, \infty]$ is arbitrary.

Since $\operatorname{spec}_{l^r} \operatorname{Op}(a^g)$ does not depend on the underlying space, the essential spectrum $\operatorname{sp}_{\mathrm{ess}\, l^p} \operatorname{Op}(a)$ is independent of $p \in (1, \infty)$. Hence, in what follows we will omit the explicit notation of the underlying space in the spectrum and the essential spectrum.

2.4. Pseudodifference operators with analytic symbols and exponential estimates of eigenfunctions

Here we introduce the notation and recall some results from section 5.3 in [42]. For $r > 1$ let $\mathbb{K}_r$ be the annulus $\{t \in \mathbb{C} : r^{-1} < |t| < r\}$, and let $\mathbb{K}_r^n$ be the product $\mathbb{K}_r \times \cdots \times \mathbb{K}_r$ of n factors.

Definition 3. *Let $\mathcal{S}\big(N, \mathbb{K}_r^n\big)$ denote the set of all functions*

$$a : \mathbb{Z}^n \times \mathbb{K}_r^n \to \mathcal{B}(\mathbb{C}^N)$$

which are analytic with respect to t in the domain $\mathbb{K}_r^n$ and satisfy the estimates

$$|a|_k := \sum_{|\alpha| \leqslant k} \sup_{x \in \mathbb{Z}^n, t \in \mathbb{K}_r^n} \big\| \partial_t^\alpha a(x, t) \big\|_{\mathcal{B}(\mathbb{C}^N)} < \infty$$

for every non-negative integer k. With every function $a \in \mathcal{S}(N, \mathbb{K}_r^n)$, we associate a pseudodifference operator defined on vector-valued functions with finite support via (9), *and we denote the corresponding class of pseudodifference operators by $OP\mathcal{S}\big(N, \mathbb{K}_r^n\big)$.*

Definition 4. *For $r > 1$, let $\mathcal{W}\big(\mathbb{K}_r^n\big)$ denote the class of all exponential weights $w = \exp v$, where v is the restriction onto $\mathbb{Z}^n$ of a function $\tilde{v} \in C^{(1)}(\mathbb{R}^n)$ with the property that, for every point $x \in \mathbb{R}^n$ and every $j = 1, \ldots, n$,*

$$-\log r < \frac{\partial \tilde{v}(x)}{\partial x_j} < \log r. \tag{18}$$

In what follows we will denote both the function $\tilde{v}$ on $\mathbb{R}^n$ and its restriction onto $\mathbb{Z}^n$ by v. Note that it is an immediate consequence of definition 4 that if $w \in \mathcal{W}(\mathbb{K}_r^n)$, then $w^\mu \in \mathcal{W}\big(\mathbb{K}_r^n\big)$ for every $\mu \in [-1, 1]$.

Proposition 2. *Let $A := \mathrm{Op}\,(a) \in OP\mathcal{S}\big(N, \mathbb{K}_r^n\big)$ and $w \in \mathcal{W}\big(\mathbb{K}_r^n\big)$. Then the operator $A_w := wAw^{-1}$, defined on vector-valued functions with finite support, belongs to the class $OP\mathcal{S}_d(N)$, and $A_w = \mathrm{Op}_d\,(b)$ with*

$$b(x, y, t) = a(x, \mathrm{e}^{-\theta_w(x,y)} \cdot t),$$

where

$$\mathrm{e}^{-\theta_w(x,y)} \cdot t := (\mathrm{e}^{-\theta_{w,1}(x,y)} t_1, \mathrm{e}^{-\theta_{w,2}(x,y)} t_2, \ldots, \mathrm{e}^{-\theta_{w,n}(x,y)} t_n)$$

and

$$\theta_{w,j}(x, y) := \int_0^1 \frac{\partial v((1-\gamma)x + \gamma y)}{\partial x_j} \,\mathrm{d}\gamma.$$

Proposition 1 and (15) imply the following theorem.

Theorem 2. *Let $a \in \mathcal{S}\big(N, \mathbb{K}_r^n\big)$ and $w \in \mathcal{W}\big(\mathbb{K}_r^n\big)$. Then $\mathrm{Op}\,(a)$ is a bounded operator on each of the spaces $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ with $1 \leqslant p \leqslant \infty$.*

Next we consider essential spectra of pseudodifference operators on weighted spaces. Let a, A and A_w be as in proposition 2. One can easily check that for $h \in \mathbb{Z}^n$

$$V_{-h} A_w V_h = \mathrm{Op}_d(b_h) \qquad \text{with} \quad b_h(x, y, t) = a(x + h, \mathrm{e}^{-\theta_w(x+h, y+h)} \cdot t)).$$

Let now $h : \mathbb{N} \to \mathbb{Z}^n$ be a sequence tending to infinity. Then there exists a subsequence g of h such that the limit operator of A_w with respect to g exists and

$$A_w^g = \mathrm{Op}_d\,(b^g) \qquad \text{with} \quad b^g(x, y, t) = a^g(x, \mathrm{e}^{\theta_w^g(x,y)} \cdot t), \tag{19}$$

where

$$a^g(x, t) := \lim_{m \to \infty} a(x + g(m), t) \tag{20}$$

and

$$\theta_w^g(x, y) := \lim_{m \to \infty} \int_0^1 \nabla v((1-\gamma)x + \gamma y + g(m)) \,\mathrm{d}\gamma. \tag{21}$$

The limits in (20) and (21) are understood as pointwise with respect to $x, y \in \mathbb{Z}^n$ and uniform with respect to $t \in \mathbb{T}^n$.

Theorem 3. *Let $a \in \mathcal{S}(N, \mathbb{K}_r^n)$ and $w \in \mathcal{W}(\mathbb{K}_r^n)$, set $A := \mathrm{Op}\,(a)$ and $A_w := wAw^{-1}$, and consider A as operating from $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ to $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ where $p \in (1, \infty)$. Then*

$$\mathrm{sp}_{\mathrm{ess}}\mathrm{Op}(a) = \bigcup_{\mathrm{Op}_d(b^g)\in op(A_w)} \mathrm{specOp}_d(b^g)$$

with b^g as in (19).

Remark 1. The theorem states that the essential spectrum of an operator in $OP\mathcal{S}(N, \mathbb{K}_r^n)$, considered as acting on $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$, is independent of $p \in (1, \infty)$, but it can depend on the weight in general. But if the weight $w = e^v$ has the property that

$$\lim_{x\to\infty} \nabla v(x) = 0, \tag{22}$$

then the symbol a_w^g does not depend on the weight and, hence, the essential spectrum of $\mathrm{Op}\,(a) \in \mathcal{B}(l^p(\mathbb{Z}^n, \mathbb{C}^N, w))$ with $a \in \mathcal{S}(N, \mathbb{K}_r^n)$ is independent both on $p \in (1, \infty)$ and on the weight w. Important examples of weights satisfying (22) are the power weights $w(x) = (1 + |x|^2)^{s/2} = \mathrm{e}^{\frac{s}{2}\log(1+|x|^2)}$ with $s > 0$ and the subexponential weights $w(x) = \mathrm{e}^{\alpha|x|^\beta}$ where $\alpha > 0$ and $\beta \in (0, 1)$.

The next theorem provides exponential estimates of solutions of pseudodifference equations.

Theorem 4. *Let $A = Op(a) \in OP\mathcal{S}(N, \mathbb{K}_r^n)$ and $w \in \mathcal{W}(\mathbb{K}_r^n)$. Suppose that $\lim_{x\to\infty} w(x) = +\infty$ and that 0 is not in the essential spectrum of $A_{w^\mu} : l^p(\mathbb{Z}^n, \mathbb{C}^N) \to l^p(\mathbb{Z}^n, \mathbb{C}^N)$ for some $p \in (1, \infty)$ and every $\mu \in [-1, 1]$. If $u \in l^p(\mathbb{Z}^n, \mathbb{C}^N, w^{-1})$ is a solution of the equation $Au = f$ with $f \in l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$, then $u \in l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$.*

Theorem 4 has some important corollaries.

Theorem 5. *Let a, A and w be as in the previous theorem, and let λ be an eigenvalue of A which is not in the essential spectrum of $A_{w^\mu} : l^p(\mathbb{Z}^n, \mathbb{C}^N) \to l^p(\mathbb{Z}^n, \mathbb{C}^N)$ for some $p \in (1, \infty)$ and every $\mu \in [0, 1]$. Then every λ-eigenfunction belongs to $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ for every $p \in (1, \infty)$.*

Corollary 1. *Let $A = Op(a) \in OP\mathcal{S}(N, \mathbb{K}_r^n)$ and let λ be an eigenvalue of A which is not in the essential spectrum of A. Then every λ-eigenfunction $u = (u_1, \ldots, u_N)$ satisfies the sub-exponential estimate*

$$\sup|u_i(x)| \leqslant C_i\,\mathrm{e}^{-\alpha|x|^\beta}, \qquad x \in \mathbb{Z}^n, \qquad i = 1, \ldots, N \tag{23}$$

for arbitrary $\alpha > 0$ and $0 < \beta < 1$.

Proof. Let $w(x) = \mathrm{e}^{v(x)}$ where $v(x) = \alpha|x|^\beta$ with $\alpha > 0$ and $0 < \beta < 1$. Then $\lim_{x\to\infty} \nabla v(x) = 0$, whence $A^g_{w^\mu} = A^g$ for every limit operator A^g. Let λ be an eigenvalue of A which is not in the essential spectrum of A. Then λ is not in the essential spectrum of A_{w^μ} for every $\mu \in [0, 1]$. Hence, by theorem 5, every λ-eigenfunction belongs to each of the spaces $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ with $p \in (1, \infty)$. Applying the Hölder inequality we obtain estimate (23). □

We are now going to specialize these results to the context of slowly oscillating symbols and slowly oscillating weights.

Definition 5. *The symbol $a \in \mathcal{S}\big(N, \mathbb{K}_r^n\big)$ is said to be slowly oscillating if*

$$\lim_{x\to\infty} \sup_{t\in\mathbb{T}^n} \|a(x+y,t) - a(x,t)\|_{\mathcal{B}(\mathbb{C}^N)} = 0$$

for every $y \in \mathbb{Z}^n$. We write $\mathcal{S}^{sl}\big(N, \mathbb{K}_r^n\big)$ for the class of all slowly oscillating symbols and $OP\mathcal{S}^{sl}\big(N, \mathbb{K}_r^n\big)$ for the corresponding class of pseudodifference operators.

Definition 6. *The weight $w = e^v \in \mathcal{W}(\mathbb{K}_r)$ is slowly oscillating if the partial derivatives $\frac{\partial v}{\partial x_j}$ are slowly oscillating for $j = 1, \dots, n$. We denote the class of all slowly oscillating weights by $\mathcal{W}^{sl}(\mathbb{K}_r)$.*

Example 1. If $v(x) = \gamma|x|$, then $\frac{\partial v(x)}{\partial x_j} = \gamma \frac{x_j}{|x|}$ for $j = 1, \dots, n$. Thus, $w := e^v$ is in $\mathcal{W}^{sl}(\mathbb{K}_r)$ if $\gamma < r$.

The next theorem describes the structure of the limit operators of the operator $A_w = wAw^{-1}$ if $A \in OP\mathcal{S}^{sl}\big(N, \mathbb{K}_r^n\big)$ and $w \in \mathcal{W}^{sl}(\mathbb{K}_r)$.

Theorem 6. *Let $A = \mathrm{Op}\,(a) \in OP\mathcal{S}^{sl}(N, \mathbb{K}_r^n)$ and $w \in \mathcal{W}^{sl}(\mathbb{K}_r)$. Then the limit operator A_w^g of A_w with respect to the sequence g tending to infinity exists if the limits*

$$a_g(t) = \lim_{m\to\infty} a(g(m), t), \qquad \theta_w^g = \lim_{m\to\infty} (\nabla v)(g(m)) \tag{24}$$

exist. In this case, it is of the form

$$A_w^g = \mathrm{Op}\,(c_g) \qquad \textit{with} \quad c_g(x,t) = a_g\big(\theta_w^g \cdot t\big). \tag{25}$$

Consequently, if A and w are as in this theorem, then the limit operators $A_{w^\mu}^g$ are invariant with respect to shifts. This fact implies the following explicit description of their essential spectra. Let $\big\{\lambda_j\big(A_{w^\mu}^g\big)(t)\big\}_{j=1}^n$ denote the eigenvalues of the matrix $a_g\big(\theta_{w^\mu}^g \cdot t\big)$. Then

$$\mathrm{spec}_{l^p(\mathbb{Z}^n,\mathbb{C}^N)} A_{w^\mu}^g = \bigcup_{j=1}^N \big\{\lambda_j\big(A_{w^\mu}^g\big)(t) : t \in \mathbb{T}^n \text{ and } j = 1, \dots, n\big\},$$

whence

$$\mathrm{sp}_{\mathrm{ess}\, l^p(\mathbb{Z}^n,\mathbb{C}^N)} A_{w^\mu} = \bigcup_{A_{w^\mu}^g \in op(A_{w^\mu})} \bigcup_{j=1}^N \big\{\lambda_j\big(A_{w^\mu}^g\big)(t) : t \in \mathbb{T}^n \text{ and } j = 1, \dots, n\big\}.$$

3. Matrix Schrödinger operators

3.1. Essential spectrum

In this section we consider the essential spectrum and the behavior at infinity of eigenfunctions of general discrete Schrödinger operators acting on $u \in l^2(\mathbb{Z}^n, \mathbb{C}^N)$ by

$$(Hu)(x) = \sum_{k=1}^n \big(V_{e_k} - \mathrm{e}^{\mathrm{i}a_k(x)}\big)\big(V_{-e_k} - \mathrm{e}^{-\mathrm{i}a_k(x)}\big)u(x) + \Phi(x), \tag{26}$$

where $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ with the 1 standing at the kth place, the $a_k \in SO(\mathbb{Z}^n)$ are real valued, and $\Phi \in SO(\mathbb{Z}^n, \mathcal{B}(\mathbb{C}^N))$ is Hermitian. The vector $a := (a_1, \dots, a_n)$ is the discrete analog of the magnetic potential, whereas Φ can be viewed of as a discrete analog of the electric potential. Since the essential spectrum of H is independent of $p \in (1, \infty)$, we consider the case $p = 2$ only. Note that our assumptions guarantee that H is a self-adjoint operator on $l^2(\mathbb{Z}^n, \mathbb{C}^N)$.

The limit operators H_g of H are of the form

$$H_g = \sum_{k=1}^{n} \left(V_{e_k} - e^{ia_k^g} I\right)\left(V_{-e_k} - e^{-ia_k^g} I\right) + \Phi^g I$$
$$= \sum_{k=1}^{n} \left(2I - e^{-ia_k^g} V_{-e_k} - e^{ia_k^g} V_{e_k}\right) + \Phi^g I$$

with the constant functions

$$a_k^g = \lim_{m\to\infty} a_k(x + g(m)) \qquad \text{and} \qquad \Phi^g = \lim_{m\to\infty} \Phi(x + g(m)).$$

Let $U : l^2(\mathbb{Z}^n, \mathbb{C}^N) \to l^2(\mathbb{Z}^n, \mathbb{C}^N)$ be the unitary operator

$$(Uu)(x) = e^{-i\langle a^g, x\rangle} u(x), \qquad a^g = (a_1^g, \ldots, a_n^g).$$

Then

$$U^* H_g U = \sum_{k=1}^{n} \left(2I - V_{-e_k} - V_{e_k}\right) + \Phi^g.$$

Further, the operator $H_g' := U^* H_g U$ is unitarily equivalent to the operator of multiplication by the function

$$\tilde{H}_g(\psi_1, \ldots, \psi_n) := 4 \sum_{k=1}^{n} \sin^2 \frac{\psi_k}{2} + \Phi^g, \qquad \psi_k \in [0, 2\pi],$$

acting on $L^2([0, 2\pi]^n, \mathbb{C}^N)$. Hence,

$$\operatorname{spec} H_g = \operatorname{spec} H_g' = \bigcup_{j=1}^{N} [\lambda_j(\Phi^g), \lambda_j(\Phi^g) + 4n],$$

where the $\lambda_j(\Phi^g)$ refer to the eigenvalues of the matrix Φ^g. Applying formula (17) we obtain

$$\operatorname{sp}_{\text{ess}} H = \bigcup_{g} \bigcup_{j=1}^{N} [\lambda_j(\Phi^g), \lambda_j(\Phi^g) + 4n], \tag{27}$$

where the first union is taken over all sequences g for which the limit operator of H exists. Let $\lambda_j(\Phi(x))$, $j = 1, \ldots, N$, denote the eigenvalues of the matrix $\Phi(x)$. We suppose that these eigenvalues are simple for x large enough and that they are increasingly ordered,

$$\lambda_1(\Phi(x)) < \lambda_2(\Phi(x)) < \cdots < \lambda_N(\Phi(x)).$$

Then one can show that the functions $x \mapsto \lambda_j(\Phi(x))$ belong to $SO(\mathbb{Z}^n)$. Let

$$\lambda_j^{\inf} := \liminf_{x\to\infty} \lambda_j(\Phi(x)), \qquad \lambda_j^{\sup} := \limsup_{x\to\infty} \lambda_j(\Phi(x)).$$

Since the set of the partial limits of a slowly oscillating function on $\mathbb{Z}^n$ is connected for $n > 1$ (see [42], theorem 2.4.7), we conclude from (27) that

$$\operatorname{sp}_{\text{ess}} H = \bigcup_{j=1}^{N} \left[\lambda_j^{\inf}, \lambda_j^{\sup} + 4n\right] \tag{28}$$

for $n > 1$. Note that if $\lambda_j^{\sup} + 4n < \lambda_{j+1}^{\inf}$, then there is the gap $\left(\lambda_j^{\sup} + 4n, \lambda_{j+1}^{\inf}\right)$ in the essential spectrum of H.

In case $n = 1$, the set of the partial limits of a slowly oscillating function on $\mathbb{Z}$ consists of two connected components, which collect the partial limits as $x \to -\infty$ and $x \to +\infty$, respectively. Accordingly, in this case we set

$$\lambda_j^{\inf,\pm} := \liminf_{x\to\pm\infty} \lambda_j(\Phi(x)), \qquad \lambda_j^{\sup,\pm} := \limsup_{x\to\pm\infty} \lambda_j(\Phi(x))$$

and obtain

$$\mathrm{sp}_{\mathrm{ess}} H = \bigcup_{j=1}^{N} \left(\left[\lambda_j^{\inf,-}, \lambda_j^{\sup,-} + 4\right] \cup \left[\lambda_j^{\inf,+}, \lambda_j^{\sup,+} + 4\right]\right).$$

3.2. *Exponential estimates of eigenfunctions*

Our next goal is to apply theorem 4 to eigenfunctions of (discrete) eigenvalues of the operator H with slowly oscillating potentials. We will formulate the results for $n > 1$ only; for $n = 1$ the non-connectedness of the set of the partial limits requires some evident modifications. According to (28), the discrete spectrum of H is located outside the set $\mathrm{sp}_{\mathrm{ess}}\, H = \bigcup_{j=1}^{N} \left[\lambda_j^{\inf}, \lambda_j^{\sup} + 4n\right]$ if $n > 1$.

Let $\cosh^{-1} : [1, +\infty) \to [0, +\infty)$ refer to the function inverse to $\cosh : [0, +\infty) \to [1, +\infty)$, i.e.,

$$\cosh^{-1} \mu = \log(\mu + \sqrt{\mu^2 - 1}).$$

Further let $\mathcal{R}^{sl} := \bigcup_{r>1} \mathcal{W}(\,\mathbb{K}_r^n)$.

Theorem 7. *Let $w = e^v$ be a weight in $\mathcal{R}^{sl}$ with $\lim_{x\to\infty} v(x) = \infty$. Further let λ be an eigenvalue of H such that $\lambda \notin \mathrm{sp}_{\mathrm{ess}}\, H$ and assume that one of the following conditions is satisfied:*

(i) there is a $j \in \{1, \dots, N\}$ such that $\lambda \in \left(\lambda_j^{\sup} + 4n, \lambda_{j+1}^{\inf}\right)$ and

$$\limsup_{x\to\infty} \left|\frac{\partial v(x)}{\partial x_k}\right| < \cosh^{-1}\left(\frac{\min\left\{\lambda - \lambda_j^{\sup} - 2n, \lambda_{j+1}^{\inf} - \lambda + 2n\right\}}{2n}\right) \tag{29}$$

for every $k = 1, \dots, n$;

(ii) $\lambda > \lambda_N^{\sup} + 4n$ and

$$\limsup_{x\to\infty} \left|\frac{\partial v(x)}{\partial x_k}\right| < \cosh^{-1}\left(\frac{\lambda - \lambda_N^{\sup} - 2n}{2n}\right)$$

for every $k = 1, \dots, n$;

(iii) $\lambda < \lambda_1^{\inf}$ and

$$\limsup_{x\to\infty} \left|\frac{\partial v(x)}{\partial x_k}\right| < \cosh^{-1}\left(\frac{\lambda_1^{\inf} - \lambda + 2n}{2n}\right)$$

for every $k = 1, \dots, n$.

Then every λ-eigenfunction of H belongs to each of the spaces $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ with $p \in (1, \infty)$.

Proof. For $\mu \in [0, 1]$, let $H'_{w^\mu} := w^\mu H' w^{-\mu}$. The limit operators $H'^g_{w^\mu} - \lambda E$ are unitarily equivalent to the operator of multiplication by the matrix function

$$\mathcal{H}^g_{w^\mu}(\psi) = \left(-2\sum_{j=1}^{n} \cos\left(\psi_j + i\mu\theta^g_j\right) + 2n - \lambda\right) E + \Phi^g$$

where

$$\psi = (\psi_1, \ldots, \psi_n) \in [0, 2\pi]^n \qquad \text{and} \qquad \theta^g_j := \lim_{m\to\infty} \frac{\partial v(g(m))}{\partial x_j}.$$

Note that

$$\Re\left(\mathcal{H}^g_{w^\mu}(\psi)\right) = \left(-2\sum_{j=1}^{n} \cos\psi_j \cosh\mu\theta^g_{w,j} + 2n - \lambda\right) E + \Phi^g,$$

where $\theta^g_{w,j} := \left(\frac{\partial v}{\partial x_j}\right)^g$. It is easy to check that condition (29) implies that $\lambda \notin \operatorname{spec} H^g_{w^\mu}$ for every limit operator $H^g_{w^\mu}$ of H_{w^μ} and every $\mu \in [0, 1]$. Hence, by theorem 5, every λ-eigenfunction belongs to $l^p(\mathbb{Z}^n, \mathbb{C}^N, w)$ for every $1 < p < \infty$. □

Corollary 2. *In each of the following cases*

(i) $\lambda \in \left(\lambda^{\sup}_j + 4n, \lambda^{\inf}_{j+1}\right)$ *for some* $j \in \{1, \ldots, N\}$ *and*

$$0 < r < \cosh^{-1}\left(\frac{\min\left\{\lambda - \lambda^{\sup}_j - 2n, \lambda^{\inf}_{j+1} - \lambda + 2n\right\}}{2n}\right),$$

(ii) $\lambda > \lambda^{\sup}_N + 4n$ *and* $0 < r < \cosh^{-1}\left(\frac{\lambda-\lambda^{\sup}_N-2n}{2n}\right)$,

(iii) $\lambda < \lambda^{\inf}_1$ *and* $0 < r < \cosh^{-1}\left(\frac{\lambda^{\inf}_1-\lambda+2n}{2n}\right)$,
every λ*-eigenfunction of* H *belongs to* $l^p(\mathbb{Z}^n, \mathbb{C}^N, e^{r|x|})$ *for each* $p \in (1, \infty)$.

Remark 2. In the case of the scalar Schrödinger operator (26) with $\Phi \in SO(\mathbb{Z}^n)$, we have

$$\mathrm{sp}_{\mathrm{ess}} H = [\Phi^{\inf}, \Phi^{\sup} + 4n]$$

with $\Phi^{\inf} = \liminf_{x\to\infty} \Phi(x)$ and $\Phi^{\sup} = \limsup_{x\to\infty} \Phi(x)$. If one of the following conditions holds for an eigenvalue λ of H:

(i) $\lambda > \Phi^{\sup} + 4n$ and $0 < r < \cosh^{-1}\left(\frac{\lambda-\Phi^{\sup}-2n}{2n}\right)$, or

(ii) $\lambda < \Phi^{\inf}$ and $0 < r < \cosh^{-1}\left(\frac{\Phi^{\inf}+2n-\lambda}{2n}\right)$, then every λ-eigenfunction of$H$ belongs to $l^p(\mathbb{Z}^n, \mathbb{C}^N, e^{r|x|})$ for each $p \in (1, \infty)$.

4. The discrete Dirac operator

4.1. The essential spectrum

On $l^2(\mathbb{Z}^3, \mathbb{C}^4)$, we consider the Dirac operators

$$\mathcal{D} := \mathcal{D}_0 + e\Phi I \qquad \text{and} \qquad \mathcal{D}_0 := c\hbar d_k\gamma^k + c^2 m\gamma^0, \tag{30}$$

where the γ^k, $k = 0, 1, 2, 3$, refer to the 4×4 Dirac matrices, i.e., they satisfy

$$\gamma^j\gamma^k + \gamma^k\gamma^j = 2\delta_{jk}E_4 \tag{31}$$

for all choices of $j, k = 0, 1, 2, 3$ where E_4 stands for the 4×4 identity matrix. Further,

$$d_k := I - V_{e_k}, \qquad k = 1, 2, 3$$

are difference operators of the first order, $\hbar$ is Planck's constant, c is the light speed, m and e are the mass and the charge of the electron, and Φ is the electric potential. We suppose that the function Φ is real valued and belongs to the space $SO(\mathbb{Z}^3)$.

It turns out that the operator $\mathcal{D}$ is *not self-adjoint* on $l^2(\mathbb{Z}^3, \mathbb{C}^4)$. Therefore we introduce self-adjoint Dirac operators as the matrix operators

$$\mathbb{D} := \mathbb{D}_0 + e\Phi I \qquad \text{with} \quad \mathbb{D}_0 := \begin{pmatrix} 0 & \mathcal{D}_0 \\ \mathcal{D}_0^* & 0 \end{pmatrix},$$

acting on the space $l^2(\mathbb{Z}^3, \mathbb{C}^8)$ (i.e., I refers now to the identity operator on that space). First we are going to determine the spectrum of $\mathbb{D}_0$. It is

$$(\mathbb{D}_0 - \lambda I)(\mathbb{D}_0 + \lambda I) = \begin{pmatrix} \mathcal{L}(\lambda) & 0 \\ 0 & \mathcal{L}(\lambda) \end{pmatrix} \tag{32}$$

where $\mathcal{L}(\lambda) = \hbar^2c^2\Gamma + (m^2c^4 - \lambda^2)I$, and

$$\Gamma := \sum_{k=1}^{3} d_k^* d_k = \sum_{k=1}^{3}(2I - V_{e_k} - V_{e_k}^*)$$

is the discrete Laplacian with symbol

$$\hat{\Gamma}(\varphi) = \hat{\Gamma}(\varphi_1, \varphi_2, \varphi_3) = \sum_{k=1}^{3}(2 - 2\cos\varphi_k), \qquad \varphi_k \in [0, 2\pi].$$

Similarly, we denote by $\hat{\mathbb{D}}_0(\varphi)$ and $\hat{\mathcal{L}}(\lambda, \varphi)$ the symbols of the operators $\mathbb{D}_0$ and $\mathcal{L}(\lambda)$, respectively. Then

$$(\hat{\mathbb{D}}_0(\varphi) - \lambda E_8)(\hat{\mathbb{D}}_0(\varphi) + \lambda E_8) = \hat{\mathcal{L}}(\lambda, \varphi)E_8 \tag{33}$$

with the scalar-valued function

$$\hat{\mathcal{L}}(\lambda, \varphi) = \hbar^2c^2\sum_{k=1}^{3}(2 - 2\cos\varphi_k) + m^2c^4 - \lambda^2.$$

We claim that $\lambda \in \operatorname{spec}\mathbb{D}_0$ if and only if there exists a $\varphi_0 \in [0, 2\pi]^3$ such that $\hat{\mathcal{L}}(\lambda, \varphi_0) = 0$. Indeed, let $\lambda \in \operatorname{spec}\mathbb{D}_0$. Then there exists a $\varphi_0 \in [0, 2\pi]^3$ such that $\det(\hat{\mathbb{D}}_0(\varphi_0) - \lambda E_8) = 0$. Hence by (33) $\hat{\mathcal{L}}(\lambda, \varphi_0) = 0$. Conversely, if $\hat{\mathcal{L}}(\lambda, \varphi_0) = 0$, then it follows from (33) that

$$(\hat{\mathbb{D}}_0(\varphi_0) - \lambda E_8)(\hat{\mathbb{D}}_0(\varphi_0) + \lambda E_8) = 0.$$

Hence, $\det(\hat{\mathbb{D}}_0(\varphi_0) - \lambda E_8) = 0$, whence $\lambda \in \operatorname{spec}\mathbb{D}_0$.

Since the equation $\hat{\mathcal{L}}(\lambda, \varphi) = 0$ has two branches of solutions (spectral curves), namely

$$\lambda_\pm(\varphi) = \pm\sqrt{\hbar^2c^2\hat{\Gamma}(\varphi) + m^2c^4}, \qquad \varphi \in [0, 2\pi]^3,$$

the spectrum of $\mathbb{D}_0$ is the union

$$\operatorname{spec}\mathbb{D}_0 = \left[-\sqrt{12\hbar^2c^2 + m^2c^4}, -mc^2\right] \cup \left[mc^2, \sqrt{12\hbar^2c^2 + m^2c^4}\right].$$

Our next goal is to determine the essential spectrum of $\mathbb{D} = \mathbb{D}_0 + e\Phi I$. All limit operators of $\mathbb{D}$ are of the form $\mathbb{D}^g = \mathbb{D}_0 + e\Phi^g I$ where $\Phi^g = \lim_{j\to\infty}\Phi(g(j))$ is the partial limit of Φ

corresponding to the sequence $g : \mathbb{N} \to \mathbb{Z}^3$ tending to infinity. By what we have just seen, this gives

$$\begin{aligned}\mathrm{spec}\mathbb{D}^g &= \left[e\Phi^g - \sqrt{12\hbar^2c^2 + m^2c^4}, e\Phi^g - mc^2\right] \\ &\cup \left[e\Phi^g + mc^2, e\Phi^g + \sqrt{12\hbar^2c^2 + m^2c^4}\right].\end{aligned}$$

Since $\mathrm{sp}_{\mathrm{ess}}\,\mathbb{D} = \cup_g \mathrm{spec}\,\mathbb{D}^g$ we obtain

$$\begin{aligned}\mathrm{sp}_{\mathrm{ess}}\mathbb{D} &= \left[e\Phi^{\mathrm{inf}} - \sqrt{12\hbar^2c^2 + m^2c^4}, e\Phi^{\mathrm{sup}} - mc^2\right] \\ &\cup \left[e\Phi^{\mathrm{inf}} + mc^2, e\Phi^{\mathrm{sup}} + \sqrt{12\hbar^2c^2 + m^2c^4}\right],\end{aligned}$$

where

$$\Phi^{\mathrm{inf}} := \liminf_{x\to\infty} \Phi(x) \qquad \text{and} \qquad \Phi^{\mathrm{sup}} := \limsup_{x\to\infty} \Phi(x).$$

In particular, if $e(\Phi^{\mathrm{sup}} - \Phi^{\mathrm{inf}}) < 2mc^2$, then the interval $(e\Phi^{\mathrm{sup}} - mc^2, e\Phi^{\mathrm{inf}} + mc^2)$ is a gap in the essential spectrum of $\mathbb{D}$.

4.2. Exponential estimates of eigenfunctions

The following is the analog of theorem 7.

Theorem 8. *Let $\lambda \notin \mathrm{sp}_{\mathrm{ess}}\,\mathbb{D}$ be an eigenvalue of $\mathbb{D} : l^p(\mathbb{Z}^3, \mathbb{C}^8) \to l^p(\mathbb{Z}^3, \mathbb{C}^8)$ with $p \in (1, \infty)$. Assume further that the weight $w = e^v$ is in $\mathcal{R}^{sl}$ and that $\lim_{x\to\infty} v(x) = \infty$. If one of the conditions*

(i) $\lambda \in (e\Phi^{\mathrm{sup}} - mc^2, e\Phi^{\mathrm{inf}} + mc^2)$ and, for every $j = 1, 2, 3$,

$$\limsup_{x\to\infty} \left|\frac{\partial v(x)}{\partial x_j}\right| < \cosh^{-1}\left(\frac{m^2c^4 - \max\{(e\Phi^{\mathrm{inf}} - \lambda)^2, (e\Phi^{\mathrm{sup}} - \lambda)^2\} + 6\hbar^2c^2}{6\hbar^2c^2}\right), \tag{34}$$

(ii) $\lambda > e\Phi^{\mathrm{sup}} + \sqrt{12\hbar^2c^2 + m^2c^4}$ and, for every $j = 1, 2, 3$,

$$\limsup_{x\to\infty} \left|\frac{\partial v(x)}{\partial x_j}\right| < \cosh^{-1}\left(\frac{(e\Phi^{\mathrm{sup}} - \lambda)^2 - m^2c^4 - 6\hbar^2c^2}{6\hbar^2c^2}\right), \tag{35}$$

(iii) $\lambda < e\Phi^{\mathrm{inf}} - \sqrt{12\hbar^2c^2 + m^2c^4}$ and, for every $j = 1, 2, 3$,

$$\limsup_{x\to\infty} \left|\frac{\partial v(x)}{\partial x_j}\right| < \cosh^{-1}\left(\frac{(e\Phi^{\mathrm{inf}} - \lambda)^2 - m^2c^4 - 6\hbar^2c^2}{6\hbar^2c^2}\right), \tag{36}$$

is satisfied, then every λ-eigenfunction of the operator $\mathbb{D}$ belongs to $l^p(\mathbb{Z}^3, \mathbb{C}^8, w)$ for each $p \in (1, \infty)$.

Proof. We will prove the assertion in case condition (i) is satisfied. The other cases follow similarly. Further, since the essential spectrum of $\mathbb{D}$ and the spectra of the associated limit operators do not depend on p, we can assume that $p = 2$ in this proof.

Let condition (34) hold, and let λ be an eigenvalue in the gap $(e\Phi^{\mathrm{sup}} - mc^2, e\Phi^{\mathrm{inf}} + mc^2)$ of the essential spectrum. In order to apply theorem 4 to determine the decaying behavior of the associated eigenfunction u_λ, we need estimates of the spectrum of the limit operators $(\mathbb{D}_{w^\mu})^g$ of $\mathbb{D}_{w^\mu} := w^\mu \mathbb{D} w^{-\mu}$ for $\mu \in [0, 1]$. The limit operator $(w^\mu V_{e_k} w^{-\mu})^g$ of $w^\mu V_{e_k} w^{-\mu}$ is of the form

$$\left(w^\mu V_{e_k} w^{-\mu}\right)^g = \mathrm{e}^{-\mu(\frac{\partial v}{\partial x_k})^g} V_{e_k}.$$

Hence,

$$(\mathcal{D}_{w^\mu})^g = \sum_{k=1}^{3} c\gamma^k \left(I - \mathrm{e}^{-\mu(\frac{\partial v}{\partial x_k})^g} V_{e_k}\right) + mc^2\gamma^0 + e\Phi^g E_4, \tag{37}$$

where $\left(\frac{\partial v}{\partial x_k}\right)^g = \lim_{m\to\infty} \frac{\partial v(g(m))}{\partial x_k}$.

Let $\mathbb{D}' = \mathbb{D}_0 - e\Phi I$. The identity (37) implies that $\left(\mathbb{D}'^g_{w^\mu} - \lambda I\right)\left(\mathbb{D}'^g_{w^\mu} + \lambda I\right)$ is the diagonal matrix diag (F, F) with

$$F := \hbar^2 c^2 \Gamma^g_{w^\mu} + (m^2c^4 - (e\Phi^g - \lambda)^2)I$$

and

$$\Gamma^g_{w^\mu} = \sum_{k=1}^{3} \left(2I - \mathrm{e}^{-(\frac{\partial v}{\partial x_k})^g} V_{e_k} - \mathrm{e}^{(\frac{\partial v}{\partial x_k})^g} V^*_{e_k}\right).$$

The operator $\Gamma^g_{w^\mu}$ is unitarily equivalent to the operator of multiplication by the function

$$\hat{\Gamma}^g_{w^\mu}(\varphi) = \hat{\Gamma}^g_{w^\mu}(\varphi_1, \varphi_2, \varphi_3) = \sum_{k=1}^{3} \left(2 - 2\cos\left(\varphi_k + i\left(\frac{\partial v}{\partial x_k}\right)^g\right)\right)$$

acting on the space $L^2([0, 2\pi]^3)$. Note that

$$\Re\left(\hat{\Gamma}^g_{w^\mu}(\varphi)\right) = 6 - 2\sum_{j=1}^{3} \cos\varphi_k \cosh\left(\frac{\partial v}{\partial x_k}\right)^g.$$

Hence, and by condition (34),

$$\Re(\hbar^2c^2\hat{\Gamma}^g_{w^\mu}(\varphi) + m^2c^4 - (e\Phi^g - \lambda)^2) \neq 0 \tag{38}$$

for every sequence g defining a limit operator and for every $\mu \in [0, 1]$. The property (38) implies that $\lambda \notin \operatorname{spec} \mathbb{D}^g_{w^\mu}$ for every limit operator $\mathbb{D}^g_{w^\mu}$ and every $\mu \in [0, 1]$. By theorem 4, every λ-eigenfunction belongs to $l^p(\mathbb{Z}^3, \mathbb{C}^8, w)$ for every $p \in (1, \infty)$. □

For the important case of the symmetric weight $w(x) = \mathrm{e}^{r|x|}$, we obtain the following corollary of theorem 8.

Corollary 3. *Let λ be an eigenvalue of $\mathbb{D} : l^p(\mathbb{Z}^3, \mathbb{C}^8) \to l^p(\mathbb{Z}^3, \mathbb{C}^8)$. If one of the conditions*

(i) $\lambda \in (e\Phi^{\sup} - mc^2, e\Phi^{\inf} + mc^2)$ and

$$0 < r < \cosh^{-1}\left(\frac{m^2c^4 + 6\hbar^2c^2 - \max\{(e\Phi^{\inf} - \lambda)^2, (e\Phi^{\sup} - \lambda)^2\}}{6\hbar^2c^2}\right),$$

(ii) $\lambda > e\Phi^{\sup} + \sqrt{12\hbar^2c^2 + m^2c^4}$ and

$$0 < r < \cosh^{-1}\left(\frac{(e\Phi^{\sup} - \lambda)^2 - m^2c^4 - 6\hbar^2c^2}{6\hbar^2c^2}\right),$$

(iii) $\lambda < e\Phi^{\inf} - \sqrt{12\hbar^2c^2 + m^2c^4}$ and

$$0 < r < \cosh^{-1}\left(\frac{(e\Phi^{\inf} - \lambda)^2 - m^2c^4 - 6\hbar^2c^2}{6\hbar^2c^2}\right),$$

is satisfied, then every λ-eigenfunction of the operator $\mathbb{D}$ belongs to $l^p(\mathbb{Z}^3, \mathbb{C}^8, \mathrm{e}^{r|x|})$ for every $p \in (1, \infty)$.

5. The square-root Klein–Gordon operator

5.1. The essential spectrum

Here we consider the square-root Klein–Gordon operator on $l^2(\mathbb{Z}^n)$, that is the operator

$$K = \sqrt{c^2\hbar^2\Gamma + m^2c^4} + e\Phi,$$

where $m > 0$ is the mass of the particle, $\hbar > 0$ is Planck's constant, $c > 0$ is the light speed, $\Phi \in SO(\mathbb{Z}^n)$ a scalar potential, and

$$\Gamma = \sum_{j=1}^{n}(2I - V_{e_j} - V_{e_j}^*)$$

is the discrete Laplacian on $\mathbb{Z}^n$. The operator $K_0 := \sqrt{c^2\hbar^2\Gamma + m^2c^4}$ is understood as the pseudodifference operator with symbol

$$k(\tau) = \sqrt{c^2\hbar^2\hat{\Gamma}(\tau) + m^2c^4} \in \mathcal{S},$$

where $\hat{\Gamma}(\tau) = \sum_{j=1}^{n}\left(2 - \tau_j - \tau_j^{-1}\right)$ at $\tau = (\tau_1, \dots, \tau_n)$. Let

$$\tilde{\Gamma}(\varphi) := \hat{\Gamma}(\mathrm{e}^{\mathrm{i}\varphi}) = \sum_{j=1}^{n}(2 - 2\cos\varphi_j), \qquad \varphi = (\varphi_1, \dots, \varphi_n) \in [0, 2\pi]^n.$$

Every limit operator of K is unitarily equivalent to an operator of multiplication by a function of the form

$$\tilde{K}^g(\varphi) = \sqrt{c^2\hbar^2\tilde{\Gamma}(\varphi) + m^2c^4} + e\Phi^g \qquad \text{with} \quad \Phi^g \in \mathbb{R}$$

acting on $L^2([0, 2\pi]^n)$. Thus,

$$\operatorname{spec} K^g = \bigcup_g \left[mc^2 + e\Phi^g, \sqrt{4nc^2\hbar^2 + m^2c^4} + e\Phi^g\right],$$

where the union is taken with respect to all sequences g tending to infinity such that the partial limit $\Phi^g := \lim_{m\to\infty}\Phi(g(m))$ exists. Consequently,

$$\mathrm{sp}_{\mathrm{ess}} K = \left[mc^2 + e\Phi^{\inf}, \sqrt{4nc^2\hbar^2 + m^2c^4} + e\Phi^{\sup}\right].$$

5.2. Exponential estimates of eigenfunctions

Theorem 9. *Let λ be an eigenvalue of the square-root Klein–Gordon operator K such that $\lambda \notin \mathrm{sp}_{\mathrm{ess}} K$, and let $w = e^v$ be a weight in $\mathcal{R}^{sl}$ with $\lim_{x\to\infty} v(x) = \infty$. If one of the conditions*

(i) $\lambda > e\Phi^{\sup} + \sqrt{4n\hbar^2c^2 + m^2c^4}$ *and*

$$\limsup_{x\to\infty}\left|\frac{\partial v(x)}{\partial x_j}\right| < \cosh^{-1}\left(\frac{m^2c^4 - (e\Phi^{\sup} - \lambda)^2 + 2n\hbar^2c^2}{2n\hbar^2c^2}\right), \tag{39}$$

(ii) $\lambda < e\Phi^{\inf} - \sqrt{4n\hbar^2c^2 + m^2c^4}$ *and*

$$\limsup_{x\to\infty}\left|\frac{\partial v(x)}{\partial x_j}\right| < \cosh^{-1}\left(\frac{m^2c^4 - (e\Phi^{\inf} - \lambda)^2 + 2n\hbar^2c^2}{2n\hbar^2c^2}\right), \tag{40}$$

is satisfied, then every λ-eigenfunction of K belongs to $l^p(\mathbb{Z}^n, w)$ for every $p \in (1, \infty)$.

Proof. The proof proceeds similarly to the proof of theorem 8. It is based on the following construction. Let $w = e^v \in \mathcal{R}^{sl}$. Then the limit operator $K^g_{w^\mu}$ is unitarily equivalent to the operator of multiplication by the function

$$\tilde{K}^g_{w^\mu}(\varphi) = \sqrt{c^2\hbar^2\tilde{\Gamma}(\varphi + i(\nabla v))^g + m^2c^4} + e\Phi^g$$

acting on $L^2([0, 2\pi]^n$. Hence,

$$\begin{aligned}\mathcal{L}^g_{w^\mu}(\varphi, \lambda) &:= \left(\tilde{K}^g_{w^\mu}(\varphi) - \lambda\right)\left(\sqrt{c^2\hbar^2\tilde{\Gamma}(\varphi + \mathrm{i}(\nabla v))^g + m^2c^4} - (e\Phi^g - \lambda)\right)\\ &= c^2\hbar^2\tilde{\Gamma}(\varphi + \mathrm{i}(\nabla v))^g + m^2c^4 - (e\Phi^g - \lambda)^2,\end{aligned}$$

and

$$\Re\left(\mathcal{L}^g_{w^\mu}(\varphi, \lambda)\right) = c^2\hbar^2 \sum_{j=1}^{n} \left(2 - \cos\varphi_j \cosh\left(\frac{\partial v}{\partial x_j}\right)^g\right) + m^2c^4 - (e\Phi^g - \lambda)^2.$$

Note that $\Re(\mathcal{L}^g_{w^\mu}(\varphi, \lambda)) \neq 0$ for every λ satisfying condition (i) or (ii). Hence, $\lambda \notin \mathrm{sp}_{\mathrm{ess}}\, K_{w^\mu}$ for every $\mu \in [0, 1]$. Thus, by theorem 4, every λ-eigenfunction belongs to the space $l^p(\mathbb{Z}^n, w)$ for all $p \in (1, \infty)$. □

Specifying the weight in the previous theorem as $w(x) = \mathrm{e}^{r|x|}$, we obtain the following.

Theorem 10. *Let λ be an eigenvalue of K such that $\lambda \notin \mathrm{sp}_{\mathrm{ess}}\, K$. If one of the conditions*

(i) $\lambda > e\Phi^{\sup} + \sqrt{4n\hbar^2c^2 + m^2c^4}]$ *and*

$$0 < r < \cosh^{-1}\left(\frac{m^2c^4 - (e\Phi^{\sup} - \lambda)^2 + 2n\hbar^2c^2}{2n\hbar^2c^2}\right),$$

(ii) $\lambda < e\Phi^{\inf} - \sqrt{4n\hbar^2c^2 + m^2c^4}$ *and*

$$0 < r < \cosh^{-1}\left(\frac{m^2c^4 - (e\Phi^{\inf} - \lambda)^2 + 2n\hbar^2c^2}{2n\hbar^2c^2}\right)$$

is satisfied, then every λ-eigenfunction of K belongs to the space $l^p(\mathbb{Z}^n, \mathrm{e}^{r|x|})$ for every $p \in (1, \infty)$.

References

[1] Agmon S 1975 Spectral properties of Schrödinger operators and scattering theory *Ann. Scuola Norm. Sup. Pisa, Cl. Sci.* **4** 151–218

[2] Agmon S 1982 *Lectures on Exponential Decay of Solutions of Second Order Elliptic Equations* (Princeton: Princeton University Press)

[3] Albeverio S, Lakaev S N and Abdullaev J I 2003 On the finiteness of the discrete spectrum of four-particle lattice Schrödinger operators *Rep. Math. Phys.* **51** 43–70

[4] Albeverio S, Lakaev S N and Muminov Z I 1994 The structure of the essential spectrum of the three-particle Schrödinger operators on a lattice arXiv:math-ph/0312050

[5] Ashkroft N W and Mermin N D 1976 *Solid State Physics* (New York: Holt, Reinhart and Winston)

[6] Basu C, Roy C L, Macía E, Domínguez-Adame F and Sánchez A 1994 Localization of relativistic electrons in a one-dimensional disordered system *J. Phys. A: Math. Gen.* **27** 3285–91

[7] Bjorken S D and Drell J D 1965 *Relativistic Quantum Mechanics* (New York: McGraw-Hill)

[8] Born M and Huang K 1954 *Dynamics Theory of Crystal Lattices* (Oxford: Clarendon)

[9] Boykin T B and Klimek G 2004 The discretized Schrödinger equation and simple models for semiconductor quantum wells *Eur. J. Phys.* **25** 503–14

[10] Buzano E 2006 Super-exponential decay of solutions to differential equations in $\mathbb{R}^d$ *Operator Theory: Adv. Appl.* vol 172 pp 117–33 (Basel: Birkhäuser)

[11] Deift P 2000 *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach* (*Courant Lectures Notes Math.* vol 3) (Providence, RI: American Mathematical Society)

[12] Froese R and Herbst I 1982 Exponential bound and absence of positive eigenvalue for N-body Schrödinger operators *Commun. Math. Phys.* **87** 429–47

[13] Froese R, Herbst I, Hoffman-Ostenhof M and Hoffman-Ostenhof T 1982 L^2-exponential lower bound of the solutions of the Schrödinger equation *Commun. Math. Phys.* **87** 265–86

[14] Hislop P D and Sigal I M 1996 *Introduction to Spectral Theory with Applications to Schrödinger Operators* (New York: Springer)

[15] Jitomirskaya S Ya 1995 Anderson localization for the almost Mathieu equation II *Commun. Math. Phys.* **168** 563–70

[16] Jitomirskaya S Ya 2000 Anderson localization for the band model *Lecture Notes Math.* vol 1745 (Berlin: Springer) pp 67–79

[17] Kuchment P and Post O 2007 On the spectra of carbon nano-structures *Commun. Math. Phys.* **275** 805–26

[18] Kuchment P 2004 On some spectral problems of mathematical physics *Partial Differential Equations and Inverse Problems* (Berlin: Springer), ed C Conca, R Manasevich, G Uhlmann and M S Vogelius *(Contemp. Math.* **362)** pp 241–76

[19] Lakaev S N and Muminov Z I 2003 The asymptotics of the number of eigenvalues of a three-particle lattice Schrödinger operator *Funct. Anal. Appl.* **37** 228–31

[20] Last Y and Simon B 2006 The essential spectrum of Schrödinger, Jacobi, and CMV operators *J. D'Anal. Math.* **98** 183–220

[21] Longhy S 2006 Non-exponential decay via tunneling in tight-binding lattices and optical Zeno effect arXiv:quant-ph/0611164v1

[22] Luckiy Ya A and Rabinovich V S 1977 Pseudodifferential operators on spaces of functions of exponential behavior at infinity *Funk. Anal. Prilozh.* **4** 79–80

[23] Madelung O 1996 *Introduction to Solid-State Theory* (Berlin: Springer)

[24] Măntoiu M 2000 Weighted estimations from a conjugate operator *Lett. Math. Phys.* **51** 17–35

[25] Măntoiu M 2002 C^*-algebras, dynamical systems at infinity and the essential spectrum of generalized Schrödinger operators *J. Reine Angew. Math.* **550** 211–29

[26] Măntoiu M and Purice R 2001 A priori decay for eigenfunctions of perturbed periodic Schrödinger operators *Ann. Henri Poincare* **2** 525–51

[27] Martinez A 1996 Microlocal exponential estimates and application to tunneling *Microlocal Analysis and Spectral Theory* (*NATO ASI Series, Ser. C: Mathematical and Physical Sciences* vol 490) ed L Rodino (Dordrecht: Kluwer) pp 349–76

[28] Martinez A 2002 *An Introduction to Semiclassical and Microlocal Analysis* (New York: Springer)

[29] Mattis D C 1986 The few-body problem on a lattice *Rev. Mod. Phys.* **58** 361–79

[30] Mogilner A 1991 *Hamiltonians in Solid State Physics as Multiparticle Discrete Schrödinger Operators: Problems and Results* (*Adv. Soviet Math.* vol 5) (Providence, RI: American Mathematical Society) pp 139–82

[31] Nakamura S 1998 Agmon-type exponential decay estimates for pseudodifferential operators *J. Math. Sci. Univ. Tokyo* **5** 693–712

[32] de Oliveira C R and Prado R A 2005 Dynamical delocalization for the 1D Bernoulli discrete Dirac operator *J. Phys. A: Math. Gen.* **38** L115–9

[33] de Oliveira C R and Prado R A 2005 Spectral and localization properties for the one-dimensional Bernoulli discrete Dirac operator *J. Math. Phys.* **46** 072105 17 pp

[34] Rabinovich V S 1995 Pseudodifferential operators with analytic symbols and some of its applications *Linear Topological Spaces and Complex Analysis* vol 2 (Ankara: Metu-Tübitak) pp 79–98

[35] Rabinovich V 2002 Pseudodifferential Operators with Analytic Symbols and Estimates for Eigenfunctions of Schrödinger Operators (*Z. Anal. Anwend. (J. Anal. Appl.)* vol 21) pp 351–70

[36] Rabinovich V S 2005 *On the Essential Spectrum of Electromagnetic Schrödinger Operators* (*Contemp. Math.* vol 382) (Providence, RI: American Mathematical Society) pp 331–42

[37] Rabinovich V S and Roch S 2004 Pseudodifference operators on weighted spaces and applications to discrete Schrödinger operators *Acta Appl. Math.* **84** 55–96

[38] Rabinovich V S and Roch S 2006 The essential spectrum of Schrödinger operators on lattices *J. Phys. A: Math. Gen.* **39** 8377–94

[39] Rabinovich V S and Roch S 2007 Essential spectra of difference operators on $\mathbb{Z}^n$-periodic graphs *J. Phys. A: Math. Theor.* **40** 10109–28

[40] Rabinovich V S and Roch S 2008 Essential spectrum and exponential decay estimates of solutions of elliptic systems of partial differential equations *Georgian Math. J.* **15** 333–51

[41] Rabinovich V S and Roch S 2008 Essential spectra of pseudodifferential operators and exponential decay of their solutions. Applications to Schrödinger operators *Operator Theory: Adv. Appl.* **181** (Basel: Birkhäuser) 355–84

[42] Rabinovich V S, Roch S and Silbermann B 2004 *Limit Operators and Their Applications in Operator Theory* (Basle: Birkhäuser)

[43] Reed M and Simon B 1979 *Methods of Modern Mathematical Physics III: Scattering Theory* (New York: Academic)

[44] Reed M and Simon B 1978 *Methods of Modern Mathematical Physics IV: Analysis of Operators* (New York: Academic)

[45] Roy C L 1994 Some special features of relativistic tunnelling through multi-barrier systems with -function barriers *Phys. Lett.* A **189** 345–50

[46] Roy C L and Basu C 1992 Relativistic study of electrical conduction in disordered systems *Phys. Rev.* B **45** 14293–301

[47] Shubin M A 1994 Discrete magnetic Laplacian *Commun. Math. Phys.* **164** 259–75

[48] Shubin M A 2001 *Pseudodifferential Operators and Spectral Theory* (Berlin: Springer)

[49] Sinai Ya 1987 Anderson localization for one-dimensional difference Schrödinger operator with quasi-periodic potential *J. Stat. Phys.* **46** 861–909

[50] Stahlhofen A A 1994 Supertransparent potentials for the Dirac equation *J. Phys. A: Math. Gen.* **27** 8279–90

[51] Thaller B 1991 *The Dirac Equation* (Berlin: Springer)

[52] Taylor M E 1981 *Pseudodifferential Operators* (Princeton, NJ: Princeton University Press)

[53] Teschl G 1999 *Jacobi Operators and Completely Integrable Nonlinear Lattices Mathematical Surveys Monogr.* vol 72 (Providence, RI: American Mathematical Society)

[54] Yafaev D R 2000 *Scattering Theory: Some Old and New Problems* (*Lecture Notes Math.* vol 1735) (Berlin: Springer)